Grade 7/8 Math Circles

November 6/7/8/9

Geometric Sequences

1. Another type of sequence are arithmetic sequences. Instead of multiplying each term by a common ratio to go from one term to the next, in an arithmetic sequence we add a **common difference** to go from one term to the next.

   Example

   (i) $\{2, 4, 6, 8, 10, 12, ...\}$ is an infinite arithmetic sequence with common difference 2.

   (ii) $\{5, 4.5, 4, 3.5, 3, 2.5\}$ is a finite arithmetic sequence with common difference -0.5.

   (iii) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}$ is not an arithmetic sequence.

   Classify the following sequences as either geometric, arithmetic, both, or neither. If it is geometric, then state the common ratio. If it is arithmetic, state the common difference.

   (a) $\{5, 10, 15, 20, 25, ...\}$
   (b) $\{1, 4, 16, 64, 256, ...\}$
   (c) $\{3, 0, -3, -6, -9\}$
   (d) $\{1, 5, 1, 5, 1, 5, 1, 5, ...\}$
   (e) $\{2, 2, 2, 2, 2, 2, ...\}$
   (f) $\{2, 7, -1, 6, 0, 3\}$

2. For each sequence in question 1, find the sum if the sequence is finite, or the sum of the first 5 terms if the sequence is infinite.

3. If we have that the first term in a geometric sequence is 5, and the common ratio is -2, write out the first 5 terms in the sequence.

   If we instead have an arithmetic sequence that has first term 5 and common difference -2, write out the first 5 terms in the sequence.

4. Suppose we have the following geometric sequence

   $$\left\{\frac{3}{2}, a, 6, 12, 24, b, 96\right\}$$

   Find a and b.

5. For each geometric sequence, identify a, r, and n. Then, using the formula for a geometric sum, find the sum of all the terms in the sequence.

   (a) $\left\{\frac{4}{2}, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}\right\}$
   (b) $\left\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\right\}$

(c) $\{-1, 3, -9, 27, -81\}$

(d) $\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\right\}$

6. Consider the fraction $\frac{1}{3}$ which we know has an infinite decimal representation,

$$\frac{1}{3} = 0.\dot{3} = 0.333333...$$

(a) Turn the infinite decimal representation into a geometric series in the same way we did for $0.\dot{9}$

(b) Identify a, r, and n for the series you created in part (a).

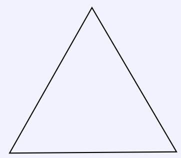
(c) Verify that, using our formula for a geometric series, the series that you created in (a) does add to $\frac{1}{3}$ like we would expect.

Extension: Sierpinski Triangle

We'll begin this section with an activity much like the one we started the lesson with.

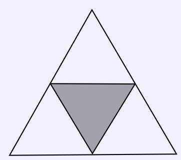
Exercise

(i) Begin by drawing an equilateral triangle.



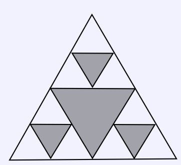
Let us pretend that this triangle has an area of 1.

(ii) Find the midpoint on all three sides and connect them to create four new triangles. Shade in the centre triangle. In fact, all four of the new triangles have the same area.



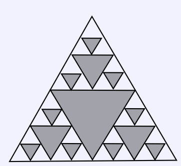
What is the total area of the unshaded regions of the triangle?

(iii) For each of the unshaded triangles, perform step (ii) again. That is, find the midpoint on all three sides and connect them to create four new triangles within the smaller triangle. Shade in the centre triangle.



What is the total area of the unshaded regions of the triangle?

(iv) Once again, repeat step (ii) for each unshaded triangle.



What is the total area of the unshaded regions?

If all went well, the areas of the unshaded regions that you've found at each step should form a geometric sequence (if not, then take a look at the solutions).

Just like with our square in the lesson, it would be quite difficult to continue dividing our triangle into smaller pieces.

However, if we were to continue dividing our triangle forever, then we would get what's known as a *Sierpinski Triangle*. You can read more about the Sierpinski Triangle by visiting this link, however much of the information here will look quite strange: https://en.wikipedia.org/wiki/Sierpi%C5%84ski_triangle

Using what you learned in the lesson, answer the following questions.

1. Similar to our formula for the unshaded regions of the square from the lesson, create a formula for the unshaded regions of the Sierpinski Triangle.
2. If we were to continue dividing the triangle, what will the area of the triangle be after
   (a) 4 divisions?
   (b) 8 divisions?
   (c) ∞ divisions?

   After an infinite number of divisions, what will the area of the shaded regions of the triangle be?

The Sierpinski Triangle is an example of something called a *fractal*. Fractals are objects that appear to look similar no matter how much you zoom in. There are many interesting looking fractal patterns, and you can find some more on the Wikipedia page for fractals, however, looking beyond the images here is not for those with weak stomachs: https://en.wikipedia.org/wiki/Fractal